Question

Prove that, $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+..............+\left[x+\frac{n-1}{n}\right]=[nx]$

Where, [ ] denotes greatest integer function.

Solution

$Let,\ x=I+f$ where I is the greatest integer part of x and f is the fractional part of x.

We know that, $\left[Integer+(.....)\right]=Integer+[(.....)]$

$LHS=[I+f]+\left[I+f+\frac{1}{n}\right]+\left[I+f+\frac{2}{n}\right]+..............+\left[I+f+\frac{n-1}{n}\right]$

$=nI+[f]+\left[f+\frac{1}{n}\right]+\left[f+\frac{2}{n}\right]+..............+\left[f+\frac{n-1}{n}\right]$

$=nI+[f]+\left[f+\frac{1}{n}\right]+\left[f+\frac{2}{n}\right]+........+\left[f+\frac{p}{n}\right]+...........+\left[f+\frac{n-1}{n}\right]$, where p is min. value such that $\left[f+\frac{p}{n}\right]=1$

$=nI+0+0+0+........p$ times$+\left[f+\frac{p}{n}\right]+...........+\left[f+\frac{n-1}{n}\right]$

$=nI+1+1+1+1+..........n-p$ times

$=nI+n-p$

$$=nI+[nf]\quad \begin{cases} \because \left[f+\frac{p}{n}\right]=1 \\ \Rightarrow \left[f+\frac{p}{n}-1\right]=0 \\ \Rightarrow \left[f-\frac{n-p}{n}\right]=0 \\ \Rightarrow f=\frac{n-p}{n}+\left(<\frac{1}{n}\right) \text{for p to be min.} \\ \Rightarrow nf=n-p+(<1) \\ \Rightarrow [nf]=n-p \end{cases}$$

$=[nI]+[nf]=[nI+nf]=[nx]$