Let $f : \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)\,dx$ lies in the interval

(A) $(2e-1, 2e)$

(B) $(e-1, 2e-1)$

(C) $\left(\frac{e-1}{2}, e-1\right)$

(D) $\left(0, \frac{e-1}{2}\right)$

Solution

We have, $f'(x) < 2f(x)$

$\therefore \frac{f'(x)}{f(x)} < 2$ $\quad [\because f(x) > 0]$

$\int_{1/2}^{x} \frac{f'(x)}{f(x)} dx < \int_{1/2}^{x} 2dx$ $\quad$ [x > 0]

$\ln f(x)\Big|_{1/2}^{x} < 2x\Big|_{1/2}^{x}$

$\Rightarrow \ln f(x) - \ln f\left(\frac{1}{2}\right) < 2\left(x - \frac{1}{2}\right)$

$\Rightarrow \ln f(x) < 2x - 1$ $\quad \left[\because f\left(\frac{1}{2}\right) = 1\right]$

$\Rightarrow f(x) < e^{2x-1}$

Also, it is given that f(x) > 0.

So, $0 < f(x) < e^{2x-1}$

$\int_{1/2}^{1} 0dx < \int_{1/2}^{1} f(x)dx < \int_{1/2}^{1} e^{2x-1}dx$

$0 < \int_{1/2}^{1} f(x)dx < \frac{e-1}{2}$

Hence, D.