Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=\begin{cases}[x], & x\le 2\\ 0, & x>2\end{cases}$,

where $[x]$ is the greatest integer less than or equal to x. If $I=\int_{-1}^{2}\frac{xf(x^2)}{2+f(x+1)}dx$, then the value of $(4I-1)$ is

Solution

$$f(x^2)=\begin{cases}[x^2], x^2\le 2\\ 0, x^2>2\end{cases}=\begin{cases}0, x>\sqrt{2}\\ [x^2], -\sqrt{2}\le x\le\sqrt{2}\\ 0, x<-\sqrt{2}\end{cases}$$

$$I=\underbrace{\int_{-1}^{1}\frac{xf(x^2)}{2+f(x+1)}dx}_{I_1}+\underbrace{\int_{1}^{\sqrt{2}}\frac{xf(x^2)}{2+f(x+1)}dx}_{I_2}+\underbrace{\int_{\sqrt{2}}^{2}\frac{xf(x^2)}{2+f(x+1)}dx}_{I_3}$$

When x lies between -1 & 1, $f(x^2)=[x^2]=0$. $\therefore I_1=0$

When x lies between $\sqrt{2}$ & 2, $f(x^2)=0$. $\therefore I_3=0$

$$\therefore I=\int_{1}^{\sqrt{2}}\frac{xf(x^2)}{2+f(x+1)}dx$$

$$f(x+1)=\begin{cases}[x+1], x+1\le 2\\ 0, x+1>2\end{cases}=\begin{cases}1+[x], x\le 1\\ 0, x>1\end{cases}$$

When x lies between 1 & $\sqrt{2}$, f(x+1) =0 & $f(x^2)=[x^2]=1$

$$\therefore I=\int_{1}^{\sqrt{2}}\frac{x.1}{2+0}dx=\frac{1}{2}.\frac{x^2}{2}\Big|_{1}^{\sqrt{2}}=\frac{1}{4}$$

$$\therefore 4I-1=0$$