Question

A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a :

(1) straight line parallel to y-axis.

(2) circle of radius 2.

(3) circle of radius $\sqrt{2}$.

(4) straight line parallel to x-axis.

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Solution

Given, $\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$

Or, $\frac{|z_1 - 2z_2|}{|2 - z_1\bar{z}_2|} = 1$

$\therefore |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\therefore (z_1 - 2z_2)\overline{(z_1 - 2z_2)} = (2 - z_1\bar{z}_2)\overline{(2 - z_1\bar{z}_2)}$

$\therefore (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1\bar{\bar{z}}_2)$

$\therefore (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$\therefore z_1\bar{z}_1 - 2z_1\bar{z}_2 - 2z_2\bar{z}_1 + 4z_2\bar{z}_2 = 4 - 2z_2\bar{z}_1 - 2z_1\bar{z}_2 + z_1 z_2\bar{z}_1\bar{z}_2$

$\therefore |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2 |z_2|^2$

$\therefore |z_1|^2\left(1 - |z_2|^2\right) - 4\left(1 - |z_2|^2\right) = 0$

$\therefore \left(|z_1|^2 - 4\right)\left(1 - |z_2|^2\right) = 0$

$\therefore |z_1| = 2$, Given $|z_2| \neq 1$

Hence, Option (2).