Question

Let y(x) be the solution of the differential equation $(x\log x)\frac{dy}{dx}+y=2x\log x,(x\ge 1)$.
Then y(e) is equal to:

(1) 0 (2) 2 (3) 2e (4) e

Solution

The given equation, $\frac{dy}{dx}+\frac{1}{x\ln x}y=2$ is linear equation of the 1st order whose solution is given by:

$$y.e^{\int\frac{1}{x\ln x}dx}=\int 2.e^{\int\frac{1}{x\ln x}dx}dx+C$$

$$\int\frac{1}{x\ln x}dx=\int\frac{\left(\frac{1}{x}\right)}{\ln x}dx=\ln\ln x \qquad \left[\because \int\frac{g'(x)}{g(x)}dx=\ln\{g(x)\}\right]$$

$$e^{\int\frac{1}{x\ln x}dx}=e^{\ln\ln x}=\ln x$$

$$\therefore y.\ln x=\int 2.\ln x dx+C$$

$$\therefore y.\ln x=2\left(\ln x.x-\int\frac{1}{x}.xdx\right)+C$$

$$\therefore y.\ln x=2x(\ln x-1)+C$$

Putting x = 1 in the original differential equation gives y = 0.

$$\therefore C=2$$

$$\therefore y.\ln x=2x(\ln x-1)+2$$

$$\therefore y=2 \text{ when } x=e$$

Hence, Option (2).