Prove that, $\sqrt{n+1}-\sqrt{n}<\dfrac{1}{2N}$ if $n>N^2$, n being a natural number.

Solution

Let $f(x)=\sqrt{x}, x\in[n,n+1]$

Since n is a natural number, clearly f(x) satisfies the continuity and differentiability requirements of LMVT.

$$f'(c)=\frac{1}{2\sqrt{c}}=\frac{f(n+1)-f(n)}{(n+1)-n}, n<c<n+1$$

$$\frac{1}{2\sqrt{c}}=f(n+1)-f(n)=\sqrt{n+1}-\sqrt{n}.........(*)$$

Since, $n<c<n+1$

$$\sqrt{n}<\sqrt{c}<\sqrt{n+1}$$

$$\frac{1}{\sqrt{n}}>\frac{1}{\sqrt{c}}>\frac{1}{\sqrt{n+1}}..........(\#)$$

Given, $n>N^2$

$$\sqrt{n}>N$$

$$\frac{1}{\sqrt{n}}<\frac{1}{N}, \frac{1}{\sqrt{n}}>\frac{1}{\sqrt{c}} \qquad \text{from } (\#)$$

Hence, $\dfrac{1}{N}>\dfrac{1}{\sqrt{c}}$

$$\frac{1}{2N}>\frac{1}{2\sqrt{c}}$$

$$\frac{1}{2N}>\sqrt{n+1}-\sqrt{n} \qquad \text{from } (*)$$