Consider the expression $S = \frac{Cos\left(\frac{B-C}{2}\right)}{Cos\left(\frac{B+C}{2}\right)} + \frac{Cos\left(\frac{C-A}{2}\right)}{Cos\left(\frac{C+A}{2}\right)} + \frac{Cos\left(\frac{A-B}{2}\right)}{Cos\left(\frac{A+B}{2}\right)}$ where A, B and C are angles of a triangle. More than one option may be correct.

(A) For right triangle, S < 6
(B) For scalene triangle, S > 6
(C) For obtuse triangle, S < 6
(D) For equilateral triangle, S = 6

Solution

We have,

$$S = \sum \frac{Cos\left(\frac{B-C}{2}\right)}{Cos\left(\frac{B+C}{2}\right)}$$

$$S = \sum \frac{Cos\left(\frac{B-C}{2}\right)}{Cos\left(\frac{\pi-A}{2}\right)}$$

$$S = \sum \frac{Cos\left(\frac{B-C}{2}\right)}{Sin\left(\frac{A}{2}\right)}$$

$$S = \sum \frac{\left[2Cos\left(\frac{A}{2}\right)\right] \times Cos\left(\frac{B-C}{2}\right)}{\left[2Cos\left(\frac{A}{2}\right)\right] \times Sin\left(\frac{A}{2}\right)}$$

$$S = \sum \frac{2Sin\left(\frac{B+C}{2}\right)Cos\left(\frac{B-C}{2}\right)}{SinA}$$

$$S = \sum \frac{SinB + SinC}{SinA}$$

$$S = \sum \frac{SinB}{SinA} + \frac{SinC}{SinA}$$

$$S = \left(\frac{SinB}{SinA} + \frac{SinC}{SinA}\right) + \left(\frac{SinC}{SinB} + \frac{SinA}{SinB}\right) + \left(\frac{SinA}{SinC} + \frac{SinB}{SinC}\right)$$

$$S = \left(\frac{SinB}{SinA} + \frac{SinA}{SinB}\right) + \left(\frac{SinC}{SinB} + \frac{SinB}{SinC}\right) + \left(\frac{SinA}{SinC} + \frac{SinC}{SinA}\right)$$

Using $A.M. \geq G.M.$ for positive numbers, $\frac{SinB}{SinA} + \frac{SinA}{SinB} \geq 2$

Similarly, $\frac{SinC}{SinB} + \frac{SinB}{SinC} \geq 2$ & $\frac{SinA}{SinC} + \frac{SinC}{SinA} \geq 2$

$\therefore S \geq 2 + 2 + 2$ Or, $S \geq 6$

The equality sign holds good only when A = B = C.

Hence, (B) & (D).